Mathematisches Institut | Universität zu Köln
Prof. Dr. Sabatini | 17.04.2019

Algebraic Topology – Homework 3

Due date : April 24th in class

Exercise 8. (12+4+4 Points)

In this exercise you will prove the so called '**Five Lemma**'. Suppose that the following diagram of abelian groups commutes :

$$\begin{array}{ccccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D & \xrightarrow{\delta} & E \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ A' & \xrightarrow[\alpha']{} & B' & \xrightarrow[\beta']{} & C' & \xrightarrow[\gamma']{} & D' & \xrightarrow[\delta']{} & E' \end{array}$$

The Five Lemma asserts that if the rows are exact, and f, g, i, j are isomorphisms of groups, then h is as well.

(i) Prove the Five Lemma.
(ii) What are the *minimal* conditions needed on f, g, i, j that ensure h to be surjective ?
(iii) What are the *minimal* conditions needed on f, g, i, j that ensure h to be injective ?

Exercise 9. (8 Points)

For an exact sequence $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E$ show that $C = 0$ if and only if the map $A \longrightarrow B$ is surjective and $D \longrightarrow E$ is injective.

Exercise 10. (8+8 Points)

(i) Compute the first singular homology group of $S^2 \setminus \{p_1, \dots, p_n\}$, where $p_1, \dots, p_n$ are pairwise different points in the 2-dimensional sphere S^2 .
(ii) Compute the first singular homology group of $\mathbb{R}^3 \setminus \{l_1, \dots, l_n\}$, where $l_1, \dots, l_n$ are pairwise different lines in $\mathbb{R}^3$ passing through the origin.

Exercise 11. (6 Points)

We denote by $\mathbb{RP}^n$ the n-dimensional real projective space and by $\mathbb{CP}^n$ the n-dimensional complex projective space. Compute the singular homology groups $H_1(\mathbb{RP}^n)$ and $H_1(\mathbb{CP}^n)$.