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Symplectic Geometry – Homework 2

Due on April 27th 2015, in class

Exercise 1.

Let $A \in GL(m;\mathbb{R})$, and consider its (unique) polar decomposition $A = P \cdot Q$, with P positive definite and symmetric and Q orthogonal. Prove that $P = \sqrt{AA^T}$ and hence $Q = (\sqrt{AA^T})^{-1}A$; namely, prove that $\sqrt{AA^T}$ is positive definite and symmetric, and $(\sqrt{AA^T})^{-1}A$ is orthogonal.

Exercise 2.

Prove that the following inclusions

$$O(2n)/U(n) \hookrightarrow GL(2n;\mathbb{R})/GL(n;\mathbb{C})$$

and

$$O(2n)/U(n) \hookrightarrow GL(2n;\mathbb{R})/Sp(2n)$$

induce a homotopy equivalence.

Exercise 3.

Let $\Omega(\mathbb{R}^{2n})$ be the space of different symplectic structures that can be assigned to $\mathbb{R}^{2n}$, and ω_0 the standard symplectic form on $\mathbb{R}^{2n}$. Then there is a natural *action* of $GL(2n;\mathbb{R})$ on $\Omega(\mathbb{R}^{2n})$ given by $(A\star\omega)(\cdot,\cdot) = A^*\omega(\cdot,\cdot) = \omega(A\cdot, A\cdot)$ for every $\omega \in \Omega(\mathbb{R}^{2n})$. Being an action means that $Id\star\omega = \omega$ and $(A\cdot B)\star\omega = A\star(B\star\omega)$ for every $\omega \in \Omega(\mathbb{R}^{2n})$.

- Prove that
$$\Omega(\mathbb{R}^{2n}) \simeq GL(2n;\mathbb{R})/Sp(2n)\,.$$
(Hints : What is the orbit of ω_0 ? And why ? What is the stabiliser of ω_0 ?)
- For which values of n does $\Omega(\mathbb{R}^{2n})$ inherit the structure of a group ?
- Compute these groups for such values of n explicitly.

Exercise 4.

(**Linear symplectic reduction**)
Let (V, ω) be a symplectic vector space and $W \subseteq V$ a coisotropic subspace. Show that the quotient W/W^{ω} carries a natural symplectic structure ω' induced by ω.

Exercise 5.

Let (V, ω) be a symplectic vector space and $W \subseteq V$ any subspace. Show that the quotient $W/(W \cap W^{\omega})$ carries a natural symplectic structure induced by ω.