Aufgabe 1.

a) I will calculate the Christoffel symbols, but the students don't have to.

$$\varphi(\theta,\alpha) = \begin{pmatrix} \sin\theta\cos\alpha \\ \sin\theta\sin\alpha \\ \cos\theta \end{pmatrix}$$

$$\Gamma^i_{jk} = \sum_l \frac{1}{2} g^{il}\left(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}\right).$$

$$\frac{\partial\varphi}{\partial\theta} = \begin{pmatrix} \cos\theta\cos\alpha \\ \cos\theta\sin\alpha \\ -\sin\theta \end{pmatrix} \qquad \frac{\partial\varphi}{\partial\alpha} = \begin{pmatrix} -\sin\theta\sin\alpha \\ \sin\theta\cos\alpha \\ 0 \end{pmatrix}$$

$$g_{ij} = \begin{pmatrix} \langle\frac{\partial\varphi}{\partial\theta},\frac{\partial\varphi}{\partial\theta}\rangle & \langle\frac{\partial\varphi}{\partial\alpha},\frac{\partial\varphi}{\partial\theta}\rangle \\ \langle\frac{\partial\varphi}{\partial\theta},\frac{\partial\varphi}{\partial\alpha}\rangle & \langle\frac{\partial\varphi}{\partial\alpha},\frac{\partial\varphi}{\partial\alpha}\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

$$\cancel{\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta}\left(\partial_\theta g_{\theta\theta}\right) = 0\,\frac{1}{2\sin^2\theta}\cdot 2\sin\theta\cos\theta = \cot\theta.}$$

$$\cancel{\Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = \frac{1}{2} g^{\theta\theta}\left(\partial_\varphi g_{\theta\theta} + \partial_\theta g_{\varphi\theta} - \partial_\theta g_{\varphi\theta}\right) = 0.}$$

$$\cancel{\Gamma^\theta_{\varphi\varphi} = \frac{1}{2} g^{\theta\theta}\left(-\partial_\theta g_{\varphi\varphi}\right) = \frac{-1}{2\sin^2\theta}\, 2\sin\theta\cos\theta = \cot\theta}$$

$$\Gamma^\theta_{\theta\theta} = 0\,\frac{1}{2} g^{\theta\theta}\left(\partial_\theta g_{\theta\theta}\right) = 0$$

$$\Gamma^\theta_{\theta\alpha} = \Gamma^\theta_{\alpha\theta} = \frac{1}{2} g^{\theta\theta}\left(+\partial_\alpha g_{\theta\theta} + \partial_\theta g_{\theta\alpha} - \partial_\theta g_{\theta\alpha}\right) = 0$$

$$\Gamma^\theta_{\alpha\alpha} = \frac{1}{2} g^{\theta\theta}\left(-\partial_\theta g_{\alpha\alpha}\right) = -\frac{1}{2}\, 2\cdot\sin\theta\cos\theta = -\sin\theta\cos\theta$$

$$\Gamma^\alpha_{\theta\theta} = \frac{1}{2} g^{\alpha\alpha}\left(\cdot\partial_\varphi g_{\theta\theta}\right) = 0$$

$$\Gamma^\alpha_{\theta\alpha} = \Gamma^\alpha_{\alpha\theta} = \frac{1}{2} g^{\alpha\alpha}\left(\partial_\theta g_{\alpha\alpha}\right) = \frac{1}{2\sin^2\theta}\cdot 2\sin\theta\cos\theta = \cot\theta.$$

$$\Gamma^\alpha_{\alpha\alpha} = 0$$

Only nonzero ones are $\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$ $\quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta$.

We know that

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ki} + \sum_{m=1}^{2} \left(\Gamma^l_{mi} \Gamma^m_{kj} - \Gamma^l_{mj} \Gamma^m_{ki} \right)$$

Moreover $R^l_{ijk} = -R^l_{jik}$, so $R^l_{iik} = 0$ always. We compute the others.
We use that $\partial_\varphi \Gamma^{\cdot}_{\cdot\cdot} = 0$, so we don't write it.

$$R^\theta{}_{\theta\varphi\theta} = -R^\theta{}_{\varphi\theta\theta} = \underbrace{\partial_\theta \Gamma^\theta_{\varphi\theta}}_{0} + \sum_m \underbrace{\Gamma^\theta_{m\theta}}_{0} \Gamma^m_{\theta\varphi} + \sum_m -\Gamma^\theta_{m\varphi} \underbrace{\Gamma^m_{\theta\theta}}_{0} = 0$$

$$R^\theta{}_{\theta\varphi\varphi} = -R^\theta{}_{\varphi\theta\varphi} = \partial_\theta \Gamma^\theta_{\varphi\varphi} + \sum_{m=1}^{2} \underbrace{\Gamma^\theta_{m\theta}}_{0} \Gamma^m_{\varphi\varphi} - \sum_{m=1}^{2} \Gamma^\theta_{m\varphi} \underbrace{\Gamma^m_{\theta\varphi}}_{0}$$

$$= -\partial_\theta(\sin\theta\cos\theta) = -\cos^2\theta + \sin^2\theta.$$

$$R^\theta{}_{\varphi\varphi\theta} = R^\theta{}_{\varphi\varphi\theta} = R^\theta{}_{\theta\theta\varphi} = R^\theta{}_{\varphi\varphi\varphi} = 0.$$

$$R^\varphi{}_{\theta\varphi\theta} = \partial_\theta \Gamma^\varphi_{\varphi\theta} + \Gamma^\varphi_{\varphi\theta} \Gamma^\varphi_{\theta\varphi} - \Gamma^\varphi_{\theta\varphi} \overset{0}{\Gamma^\theta_{\theta\theta}}$$

$$= \partial_\theta \cot\theta + \cot^2\theta.$$

$$= \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} + \cot^2\theta = -1. = -R^\varphi{}_{\varphi\theta\theta}.$$

$$\Gamma^\varphi{}_{\theta\varphi\varphi} = \underbrace{\partial_\theta \Gamma^\varphi_{\varphi\varphi}}_{0} - 0 + \Gamma^\varphi_{\varphi\theta} \underbrace{\Gamma^\varphi_{\varphi\varphi}}_{0} - \Gamma^\varphi_{\theta\varphi} \overset{0}{\Gamma^\theta_{\varphi\theta}} = 0$$
$$= -\Gamma^\varphi_{\varphi\theta\varphi}$$

$$\Gamma^\varphi{}_{\varphi\theta\theta} = \Gamma^\varphi{}_{\varphi\varphi\theta} = \Gamma^\varphi{}_{\theta\theta\varphi} = \Gamma^\varphi{}_{\varphi\varphi\varphi} = 0$$

5a. b). $\cancel{I}(R_p(v,w)\,w, \cancel{v}) = R^\alpha{}_{\alpha\theta\theta} \frac{\partial\varphi}{\partial\alpha} = 1 \frac{\partial\varphi}{\partial\alpha}$.

$$\underline{I}(R_p(v,w)\,w, v) = 1 = K(p)$$

as $K(p) = \det W_p = 1$

A.

Exercise 2:

This follows from Lemma 4.3.4.

As $R(v,w)w = K(p)\left(I(w,w)v - I(v,w)w\right)$

we find $I(R(v,w)w, v)$

$$= K(p)\left(I(w,w)I(v,v) - I(v,w)I(w,v)\right) \quad \square.$$

Exercise 3:

$$\cancel{R_{mijk} = \sum_{l=1}^{2} g_{ml} R^{l}_{ijk}}$$

$$\cancel{= \partial_l \ldots \sum g_{k\ldots}}$$

This can be done from definitions:

$R_{mijk} + R_{mjki} + R_{mkij} =$

$$\cancel{\sum_{l=1}^{2}\left(g_{ml}R^{l}_{ijk} + g_{ml}R^{l}_{jki} \ldots\right)}$$

$$= \sum_{l=1}^{2} g_{ml}\left\{\left(\partial_i \Gamma^{l}_{kj} - \partial_j \Gamma^{l}_{ki}\right) + \sum_n \Gamma^{l}_{ni}\Gamma^{n}_{kj} - \Gamma^{l}_{nj}\Gamma^{n}_{ki}\right\} +$$

$$\sum_{l=1}^{2} g_{ml}\left\{\left(\partial_j \Gamma^{l}_{ik} - \partial_k \Gamma^{l}_{ij}\right) + \sum_n \Gamma^{l}_{nj}\Gamma^{n}_{ik} - \Gamma^{l}_{nk}\Gamma^{n}_{ij}\right\} +$$

$$\sum_{l=1}^{2} g_{ml}\left\{\left(\partial_k \Gamma^{l}_{ji} - \partial_i \Gamma^{l}_{jk}\right) + \sum_n \Gamma^{l}_{nk}\Gamma^{n}_{ji} - \Gamma^{l}_{ni}\Gamma^{n}_{jk}\right\}$$

$= 0$ (Same symbol drops out)

Follows of course also from 4.3.3.