Mathematisches Institut | Universität zu Köln
Prof. Dr. Sabatini | 07.06.2019

Algebraic Topology – Homework 10

Due date : June 19th in class.

Exercise 36. (10 Points)

Let A, B and C be abelian groups, and $A \xrightarrow{i} B \xrightarrow{j} C \to 0$ an exact sequence. Prove that the 'dual sequence'

$$\mathrm{Hom}(A;G) \xleftarrow{i^*} \mathrm{Hom}(B;G) \xleftarrow{j^*} \mathrm{Hom}(C;G) \leftarrow 0$$

is exact.

Exercise 37. (5+5+5 Points)

Let Σ be a compact surface endowed with its standard CW structure and $(C_\bullet^{\mathrm{CW}}(\Sigma), \partial_\bullet)$ be its corresponding cellular chain complex.

(i) Compute the cellular homology groups of Σ with $\mathbb{Z}_2$ coefficients using the chain complex $(C_\bullet^{\mathrm{CW}}(\Sigma) \otimes \mathbb{Z}_2, \partial_\bullet \otimes \mathrm{Id}_{\mathbb{Z}_2})$.

(ii) Compute the homology groups of Σ with $\mathbb{Z}_2$ coefficients using the Universal Coefficient Theorem.

(iii) Compute the cohomology groups of the cellular chain complex $(C_\bullet^{\mathrm{CW}}(\Sigma), \partial_\bullet)$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$

Note : You need to use the classification theorem of compact surfaces and make a distinction between the orientable and non-orientable case, for every genus g.

Exercise 38. (10 Points)

Compute the cohomology groups of the (standard) cellular chain complex of $\mathbb{R}\mathrm{P}^n$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$, for every $n \geq 1$. Verify your answer by using the Universal Coefficient Theorem for cohomology.

<u>Exercise</u> 39. (5+5+5 Points)

Consider the real projective space $\mathbb{R}\mathrm{P}^2$ endowed with the usual CW structure, with one cell in dimension $0, 1$ and 2, so that the one skeleton X^1 is homeomorphic to $\mathbb{R}\mathrm{P}^1$ and the (only) two cell e^2 is attached to X^1 via a map of degree 2. Let $p\colon \mathbb{R}\mathrm{P}^2 \longrightarrow \mathbb{R}\mathrm{P}^2/X^1 \simeq S^2$ be the projection map obtained by collapsing X^1 in $\mathbb{R}\mathrm{P}^2$. Compute the following maps :

(i) $(p_\# \otimes \mathrm{Id}_{\mathbb{Z}_2})_*\colon H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) \longrightarrow H_2(S^2;\mathbb{Z}_2)$;

(ii) $p_*\colon H_i(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \longrightarrow H_i(S^2;\mathbb{Z})$ for all i.

Consider now the following commutative diagram involving the *Universal Coefficient Theorem* for $H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2)$ and $H_2(S^2;\mathbb{Z}_2)$ and the homomorphisms induced by the map $p\colon \mathbb{R}\mathrm{P}^2 \longrightarrow S^2$ defined above :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\mathbb{R}\mathrm{P}^2)\otimes\mathbb{Z}_2 & \xrightarrow{\alpha_1} & H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) & \xrightarrow{\beta_1} & \mathrm{Tor}(H_1(\mathbb{R}\mathrm{P}^2),\mathbb{Z}_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_*\otimes \mathrm{Id}_{\mathbb{Z}_2}} & & \downarrow{\scriptstyle (p_\#\otimes \mathrm{Id}_{\mathbb{Z}_2})_*} & & \downarrow{\scriptstyle \mathrm{Tor}(p_*,\mathrm{Id}_{\mathbb{Z}_2})} & & \\
0 & \longrightarrow & H_2(S^2)\otimes\mathbb{Z}_2 & \xrightarrow{\alpha_2} & H_2(S^2;\mathbb{Z}_2) & \xrightarrow{\beta_2} & \mathrm{Tor}(H_1(S^2),\mathbb{Z}_2) & \longrightarrow & 0
\end{array}$$

The Universal Coefficient Theorem states that the horizontal arrows in the diagram are splitting exact sequences, meaning that there exist morphisms

$$p_1\colon H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) \longrightarrow H_2(\mathbb{R}\mathrm{P}^2)\otimes\mathbb{Z}_2$$
$$p_2\colon H_2(S^2;\mathbb{Z}_2) \longrightarrow H_2(S^2)\otimes\mathbb{Z}_2$$

such that

$$p_1\circ\alpha_1 = \mathrm{Id}_{H_2(\mathbb{R}\mathrm{P}^2)\otimes\mathbb{Z}_2}\ ,\ \ p_2\circ\alpha_2 = \mathrm{Id}_{H_2(S^2)\otimes\mathbb{Z}_2}\ .$$

(iii) Does the following diagram

$$\begin{array}{ccc}
H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) & \xrightarrow{p_1\oplus\beta_1} & (H_2(\mathbb{R}\mathrm{P}^2)\otimes\mathbb{Z}_2)\oplus\mathrm{Tor}(H_1(\mathbb{R}\mathrm{P}^2),\mathbb{Z}_2) \\
\downarrow{\scriptstyle (p_\#\otimes \mathrm{Id}_{\mathbb{Z}_2})_*} & & \downarrow{\scriptstyle (p_*\otimes \mathrm{Id}_{\mathbb{Z}_2},\mathrm{Tor}(p_*,\mathrm{Id}_{\mathbb{Z}_2}))} \\
H_2(S^2;\mathbb{Z}_2) & \xrightarrow[p_2\oplus\beta_2]{} & (H_2(S^2)\otimes\mathbb{Z}_2)\oplus\mathrm{Tor}(H_1(\mathbb{R}\mathrm{P}^2),\mathbb{Z}_2)
\end{array}$$

commute ? Can the map in (i) be recovered from the maps in (ii) and the *Universal Coefficient Theorem* for homology ?