Mathematisches Institut
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Universität zu Köln
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Algebraic Topology – Homework 10

Due date : January 7th in class

Exercise 1.

Prove the following

Lemma 1. *Let X be a topological space. Then*

(i) For any homology class $x \in H_n(X)$ there exists a compact set $C \subset X$, with inclusion map $i\colon C \longrightarrow X$, and a homology class $x' \in H_n(C)$ such that $i_*(x') = x$.

(ii) Let $y \in H_n(C)$, with $C \subset X$ compact, such that $i_*(y) = 0$. Then there exists a compact subset C' satisfying $C \subset C' \subset X$, with inclusion map $j\colon C \longrightarrow C'$, such that $j_*(y) = 0$.

Exercise 2.

Prove the following corollary of the Jordan-Brouwer separation theorem :

Corollary 2. *Let $f\colon S^{n-1} \longrightarrow \mathbb{R}^n$ be an embedding of an $(n-1)$-dim. sphere into $\mathbb{R}^n$, i.e. f is a homeomorphism onto its image, and suppose that $n \geq 2$. Then $\mathbb{R}^n \setminus f(S^{n-1})$ has two components U and V, U unbounded and V bounded, such that V is acyclic and U satisfies $H_i(U) \simeq H_i(S^{n-1})$ for every i.*

Exercise 3.

Compute the homology of $S^2 \times S^1$ using the Mayer-Vietoris sequence associated to the sets $A = S^2 \times (S^1 \setminus \{p\})$ and $B = S^2 \times (S^1 \setminus \{q\})$, where p and q are distinct points in S^1.

Exercise 4.

Exercise number 4 on page 205 of Hatcher's book. Observe that in particular you should define a boundary operator $\delta_n\colon \mathrm{Hom}(G, C_n(X)) \longrightarrow \mathrm{Hom}(G, C_{n-1}(X))$ using that of the given chain complex $(C_., \partial_.)$, and check that what you obtain is a chain complex $(\mathrm{Hom}(G, C_.), \delta_.)$.