Mathematisches Institut — Universität zu Köln
Prof. Dr. Sabatini — November 19th 2014

Algebraic Topology – Homework 6

Due date : November 26th in class

Exercise 1.

Let X be a nonempty topological space with $n < \infty$ path-connected components. Prove that $\widetilde{H}_0(X) \simeq \mathbb{Z}^{n-1}$ if $n > 1$ and $\widetilde{H}_0(X) = 0$ if $n = 1$ explicitly, by exhibiting a basis of it.

Exercise 2.

Let X, Y be topological spaces, and $f\colon X \longrightarrow Y$ a constant map. Prove that $f_*\colon H_i(X) \longrightarrow H_i(Y)$ is the zero homomorphism for every $i > 0$.

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Let $\{A_n\}_{n\in\mathbb{Z}}$ be a sequence of abelian groups, and $\{\alpha_n\colon A_{n+1} \longrightarrow A_n\}_{n\in\mathbb{Z}}$ be homomorphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

so that $\mathrm{Ker}\ \alpha_n = \mathrm{Im}\ \alpha_{n+1}$ for every n. Thus the pair $(A_*, \alpha_*) = \{(A_n, \alpha_n)\}_{n\in\mathbb{Z}}$ is a chain complex with *trivial homology*, and is called an **exact sequence**.
In particular

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is called a **short exact sequence**. This is equivalent to saying that α is *injective*, β is *surjective* and $\mathrm{Im}\,\alpha = \mathrm{Ker}\,\beta$, thus implying that $B/\,\mathrm{Im}\,\alpha \simeq C$.

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Exercise 3.

Suppose that

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

is an exact sequence. Prove that for every n there is a short exact sequence of the form

$$0 \longrightarrow \mathrm{Coker}\,\alpha_{n+2} \xrightarrow{\widetilde{\alpha}_{n+1}} A_n \xrightarrow{\alpha'_n} \mathrm{Ker}\,\alpha_{n-1} \longrightarrow 0$$

where $\mathrm{Coker}\,\alpha_i$ denotes the cokernel of α_i, i.e. it is equal to $A_{i-1}/\,\mathrm{Im}\,\alpha_i$. Note that you need to define the maps $\widetilde{\alpha}_{n+1}$ and α'_n, and prove that they are well-defined.

<u>Exercise</u> 4.

Let $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be a short exact sequence.

(a) Suppose that there exists a homomorphism $s\colon C \longrightarrow B$ such that $\beta \circ s = Id_C$. Then the short exact sequence is called **split**, and the map s a **splitting**. Show that in this case $B \simeq A \oplus C$.

(b) Suppose that $C = \mathbb{Z}^m$. Show that there always exists a splitting $s\colon \mathbb{Z}^m \longrightarrow B$. Give an example of a short exact sequence that does not admit *any* splitting.

(c) Use the argument above to prove carefully that, given a topological space X, then $H_0(X) \simeq \widetilde{H}_0(X) \oplus \mathbb{Z}$.