Mathematisches Institut — Universität zu Köln
Prof. Dr. Sabatini — December 3rd 2014

Algebraic Topology – Homework 8

Due date : December 10th in class

Exercise 1.

(Euler characteristic of a finite CW complex)
We recall that every finitely generated abelian group G can be written as a direct sum of cyclic subgroups, and the number of $\mathbb{Z}$ summands is defined to be the *rank of* G, denoted by $\operatorname{rank}(G)$. In this exercise you can use the following

Lemma 1. *Let A, B and C be finitely generated abelian groups such that the following sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact. Then $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$*.*

(a) Let C_i be finitely generated abelian groups. Prove that given a chain complex :

$$0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

with homology groups $H_i(C)$, then

$$\sum_{i=0}^{n} (-1)^i \operatorname{rank}(C_i) = \sum_{i=0}^{n} (-1)^i \operatorname{rank}(H_i(C))$$

(b) Let X be a topological space which can be endowed with the structure of a *finite* CW complex $X = X^n$, i.e. the number of i-dimensional cells, denoted by c_i, is finite for every i, and zero for $i > n$. Define the **Euler characteristic** of X to be

$$\chi(X) = \sum_{i=0}^{n} (-1)^i c_i$$

Prove that $\chi(X)$ only depends on the homotopy type of X.

Exercise 2.

Exercise 2 on page 155 of Hatcher's book. In order to solve this exercise, you will need to use that the sphere S^n is a covering space of $\mathbb{R}\mathrm{P}^n$ with respect to the projection map $a\colon S^n \longrightarrow \mathbb{R}\mathrm{P}^n$ (see Hatcher's book page 56), and hence Proposition 1.33 on page 61 holds.

Exercise 3.

Exercise 8 on page 155 of Hatcher's book. For this exercise you can use Proposition 2.30 on page 136.

Exercise 4.

Let X be a topological space endowed with the structure of a CW complex. Prove that $H_n(X^n)$ is a free abelian group for every n. (Here X^n denotes the n-skeleton of the CW complex.)